Generalized analytic functions, Moutard-type transforms and holomorphic maps

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Abstract

We continue the studies of Moutard-type transform for generalized analytic functions started in [1]. In particular, we suggest an interpretation of generalized analytic functions as spinor fields and show that in the framework of this approach Moutard-type transforms for the aforementioned functions commute with holomorphic changes of variables.

Key words: generalized analytic functions, spinors, Moutard transforms.

We study the basic pair of conjugate equations of the generalized analytic function theory:

$$\partial_{\bar z}\psi = u\bar\psi \ \text{ in } \ D, \tag{1}$$

$$\partial_{\bar z}\psi^+ = -\bar u\bar\psi^+ \ \text{ in } \ D, \tag{2}$$

where D is an open domain in $\mathbb{C}$, $u = u(z)$ is a given function in D, $\partial_{\bar z} = \partial/\partial\bar z$; see [4]. Here and below, the notation $f = f(z)$ does not mean that f is holomorphic.

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A new progress in the theory of generalized analytic functions was obtained very recently in [1] by showing that Moutard-type transforms can be applied to the pair of equations (1), (2). Note that ideas of Moutard-type transforms were developed and successfully used in the differential geometry, in the soliton theory in dimension 2+1, and in the spectral theory in dimension 2, see [1] for further references. In particular, our work [1] was essentially stimulated by recent articles by I.A. Taimanov [2], [3] on the Moutard-type transforms for the Dirac operators in the framework of the soliton theory in the dimension $2+1$. On the other hand, we were strongly motivated by some open problems of two-dimensional inverse scattering at fixed energy, where equation (1) arises as the $\bar\partial$-equation in spectral parameter.

A simple Moutard-type transform $\mathcal{M} = \mathcal{M}_{u,f,f^+}$ for the pair of conjugate equations (1), (2) is given by the formulas (see [1]):

$$\tilde u = \mathcal{M}u = u + \frac{f\overline{f^+}}{\omega_{f,f^+}}, \tag{3}$$

$$\tilde\psi = \mathcal{M}\psi = \psi - \frac{\omega_{\psi,f^+}}{\omega_{f,f^+}}\, f, \quad \tilde\psi^+ = \mathcal{M}\psi^+ = \psi^+ - \frac{\omega_{f,\psi^+}}{\omega_{f,f^+}}\, f^+, \tag{4}$$

where f and f^+ are some fixed solutions of equations (1) and (2), respectively, ψ and ψ^+ are arbitrary solutions of (1) and (2), respectively, and $\omega_{\psi,\psi^+} = \omega_{\psi,\psi^+}(z)$ denotes imaginary-valued function defined by:

$$\partial_z\omega_{\psi,\psi^+} = \psi\psi^+, \quad \partial_{\bar z}\omega_{\psi,\psi^+} = -\overline{\psi\psi^+} \ \text{ in } \ D, \tag{5}$$

where this definition is self-consistent, at least, for simply connected D, whereas a pure imaginary integration constant may depend on concrete situation. The point is that the functions $\tilde\psi$, $\tilde\psi^+$ defined in (4) satisfy the conjugate pair of Moutard-transformed equations:

$$\partial_{\bar z}\tilde\psi = \tilde u\,\overline{\tilde\psi} \qquad \text{in } \ D, \tag{6}$$

$$\partial_{\bar z}\tilde\psi^+ = -\overline{\tilde u}\,\overline{\tilde\psi^+} \qquad \text{in } \ D. \tag{7}$$

In addition, we have also the following new important result:

Proposition 1 *For a simple Moutard transform (3),(4) the following formula holds:*

$$\omega_{\tilde\psi,\tilde\psi^+} = \frac{\omega_{\psi,\psi^+}\omega_{f,f^+} - \omega_{\psi,f^+}\omega_{f,\psi^+}}{\omega_{f,f^+}} + c_{\tilde\psi,\tilde\psi^+}, \tag{8}$$

where $c_{\tilde\psi,\tilde\psi^+}$ is an imaginary constant.

In order to apply the Moutard-type transforms to studies of generalized-analytic functions with contour poles, we need to study, in particular, compositions of the former transforms and holomorphic maps.

Consider a holomorphic bijection W:

$$W : D \to D_*, \quad z \to \zeta(z), \tag{9}$$
$$W^{-1} : D_* \to D, \quad \zeta \to z(\zeta),$$

where D is the domain in (1), (2).

If we treat $\psi(z)$, $\psi^+(z)$ as scalar fields in equations (1), (2), then the conjugate property of these equations is not invariant with respect to holomorphic bijections . In the next theorem we give the proper transformation formulas for the conjugate pair of equations (1), (2) with respect to holomorphic bijections:

Theorem 1 *Let W be a holomorphic bijection as in (9). Let*

$$u_*(\zeta) = u(z(\zeta))\sqrt{\frac{\partial z}{\partial \zeta}\frac{\partial \bar{z}}{\partial \bar{\zeta}}} = u(z(\zeta))\left|\frac{\partial z}{\partial \zeta}\right|, \tag{10}$$

$$\psi_*(\zeta) = \psi(z(\zeta))\sqrt{\frac{\partial z}{\partial \zeta}}, \quad \psi_*^+(\zeta) = \psi^+(z(\zeta))\sqrt{\frac{\partial z}{\partial \zeta}}, \tag{11}$$

where $u(z)$, $\psi(z)$, $\psi^+(z)$ are the same that in equations (1), (2). Then:

$$\partial_{\bar{\zeta}}\psi_* = u_*\bar{\psi}_* \quad in \quad D_*, \tag{12}$$
$$\partial_{\bar{\zeta}}\psi_*^+ = -\bar{u}_*\bar{\psi}_*^+ \quad in \quad D_*. \tag{13}$$

In addition,

$$\omega_{\psi_*,\psi_*^+}(\zeta) = \omega_{\psi,\psi^+}(z(\zeta)), \tag{14}$$

where ω is defined according to (5).

Remark 1 *Formulas (10), (11) have the following natural interpretation: $\psi(z)$, $\psi^+(z)$ can be treated as spinors, i.e. differential forms of the type $\left(\frac{1}{2}, 0\right)$, and u can be treated as differential form of the type $\left(\frac{1}{2}, \frac{1}{2}\right)$. The corresponding forms can be written as:*

$$u = u(z)\sqrt{dzd\bar{z}}, \quad \psi = \psi(z)\sqrt{dz}, \quad \psi^+ = \psi^+(z)\sqrt{dz}. \tag{15}$$

It is very natural because the generalized analytic function equation (1) can be viewed as a special reduction of the two-dimensional Dirac system, see, for example [1].

Theorem 1 implies that W in (9) generates a map of the conjugate pair of equations (1), (2) into the conjugate pair of equations (12), (13). We also denote the latter map by W. Using this interpretation of W we obtain the following result:

Theorem 2 *The following formula holds:*

$$\mathcal{M}_{u_*,f_*,f_*^+} \circ W = W \circ \mathcal{M}_{u,f,f^+}, \tag{16}$$

where $\mathcal{M}_{u,f,f^+}$ *and* $\mathcal{M}_{u_*,f_*,f_*^+}$ *are defined according to formulas (3), (4), and* u_*, f_*, f_*^+, ω_{ψ_*,ψ_*^+} *are defined according to (10), (11), (14).*

Proposition 1 and Theorems 1 and 2 can be proved by direct calculations.

In the framework of the Moutard transform approach, using Theorem 2 we reduce local studies of generalized analytic functions with contour pole at a real-analytic curve to the case of contour pole at a straight line. These studies will be continued in a subsequent paper.

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